ON THE IMPROBABILITY OF FINDING ISOLATED SHOALS IN THE OPEN SEA BY SAILING OVER THE GEOGRAPHICAL POSITIONS IN WHICH THEY ARE CHARTED.

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Many of the isolated shoals that are represented on nautical charts of the oceans have been located from the reports of mariners who have discovered them incidentally in making voyages of commerce. Previous to the year 1860, when there was no exact knowledge of the depths of the oceans, the vague reports of navigators, often doubtless based upon the observation of floating objects and of misleading appearances of the surface of the sea, caused the charting of many dangers for the existence of which there is no substantial foundation. But, as our knowledge of bathymetry increased, the existence of many of them was disproved, and they were removed from the charts.

As a result of these experiences, there arose a traditional distrust among mariners and hydrographers of the existence of many of these dangers that still appear on the charts with well founded evidence, and there is perhaps a disposition on the part of many to claim that they should be removed upon scant evidence of their non-existence. It is not uncommon for a mariner to report that, being in the vicinity of a charted rock or shoal, he laid his course so as to pass over the geographical position assigned to it with one hundred fathoms of line out or with lookouts posted aloft, but was unable to detect any evidence of its existence, and that he does not believe, therefore, that the rock or shoal exists.

It seems necessary, therefore, to inquire into the degree of confidence that can be placed in such a piece of evidence of the non-existence of a danger, and to establish what probability there would be of finding it under these conditions.

Suppose that A discovers, in the open ocean, a shoal r miles in radius, and determines the geographical position of its center subject to extreme errors of m miles in longitude and n miles in latitude; and that B, who is able to establish his geographical position within the same limits of extreme error as A, attempts to find the shoal again by proceeding to the geographical position assigned to it by A. What is the probability that he will find it?

If A, after making the discovery, had revisited the shoal a great number of times and had deduced the latitude and longitude of the same spot, under the same circumstances, at each visit, the latitudes would all differ from the

true latitude, and, likewise, the longitude from the true longitude. If we call the differences between the true latitude and the deduced latitudes errors of latitude, and lay them off, according to their signs, to the right and left of an assumed origin, and then, corresponding to each error as an abscissa, erect an ordinate of a length proportional to the probability of that error, these ordinates and abscissas will be the coordinates of the probability curve. And, likewise, if the errors in longitude were found and plotted in conjunction with their probabilities, a similar curve would be developed.

In this investigation the probability curve, ordinarily represented by Laplace's formula, $y = ce^{-a^2t^2}$, will be replaced by two equally inclined straight lines AB and AB' as shown in figure 1.

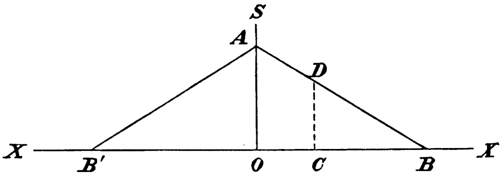

FIGURE 1.

This substitution, which has been employed by Hélie in his Traité de Balistique Expérimentale and referred to by Wright in his work on the Adjustment of Observations, causes an appreciable but extremely small error which has no practical significance when we consider that, from the nature of the calculations about to be made, absolute precision is not to be sought.

The probability of having an error between $OC = x$ and $x + \Delta x$ (figure 1) to the right of the axis OS is equal to $s\Delta x$. As, in this case, OB and OB' measure the extreme errors, all possible errors are comprised between zero and OB, and zero and OB'; and the sum of all the elements which are singly represented by $s \,.\, dx$, or the area of the triangle ABB', should be equal to unity which is the measure of certainty. The equation to the straight line AB will be, calling m the extreme error OB and b the intercept on the axis of S,

$$\frac{s}{b} + \frac{x}{m} = 1\,.$$

But, since the area $ABB' = b \times m = 1$ or $b = \dfrac{1}{m}$, this equation becomes

$$sm + \frac{x}{m} = 1\,,$$

or

$$s = \frac{m - x}{m^2}\,.$$

And since x can only vary between zero and m, the probability of having an error between x and $x + \Delta x$ will be :

$$p_1 = \frac{m - x}{m^2} \Delta x \,. \tag{1}$$

The causes which produce the grouping of a number of deduced geographical positions around the true one are of two kinds ; one tending to place the deduced latitude to the north or south of the true latitude, and the other tending to place the deduced longitude to the east or west of the true longitude. So that a particular deduced geographical position P will be the result of having an error OA in latitude and an error OB in longitude.

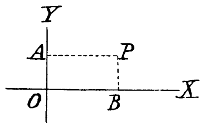

FIGURE 2.

The probability that the geographical position deduced by A, upon his discovery of the shoal, occupies a certain position with reference to the true geographical position of the shoal is, therefore, easily deduced. Through the true geographical position of the shoal let two rectangular axes, OX and OY, be passed as shown in figure 2. Upon the former conceive errors in longitude to be measured, and upon the latter, errors in latitude. The position P, of which the coordinates are x and y, results from the concurrence of two conditions, the error of x miles in longitude and the error of y miles in latitude. The probability p_1 of an error between x and $x + \Delta x$ is, as shown by equation (1),

$$p_1 = \frac{m - x}{m^2} \Delta x \,;$$

and, in the same manner, the probability p_2 of an error between y and $y + \Delta y$ will be

$$p_2 = \frac{n - y}{n^2} \Delta y \,. \tag{2}$$

In these formulas, m and n represent respectively the extreme errors in longitude and latitude in miles.

The probability p of having, at the same time, the error x and the error y, or of deducing the geographical position P as the position of the shoal, will be the product $p_1 p_2$, or

$$p = \frac{(m - x)(n - y)}{m^2 n^2} \Delta x \, \Delta y, \tag{3}$$

an equation in which x can vary from zero to m, and y from zero to n. It is, therefore, applicable to the first right angle of the axes OX and OY, but, in order to make it applicable to other quadrants, it is only necessary to change the signs of x and y.

Equation (3) then expresses the probability that A's determination of the geographical position of the shoal is in error by x miles in longitude and y miles in latitude.

If the center of the shoal were really located in the geographical position assigned to it by A, and B should succeed in coming within r miles of it, he would find the shoal since its radius is r miles.

We have, therefore, as the second step in the solution of the problem, to determine what is the probability that B will come within a circular area, r miles in radius, having its center anywhere within the rectangle described about the true position of the shoal with sides equal to the extreme errors to which the determinations of latitude and longitude by A and B are subject.

To find the probability, P, of coming within any portion of the rectangle of extreme errors inclosed by a curve whose equation is $y = f(x)$, it is sufficient to integrate the expression (3) between limits depending only upon $y = f(x)$, and we shall have, in the first right angle,

$$P = \frac{1}{m^2 n^2} \int dx \int (m - x)(n - y)\, dy\,. \tag{4}$$

For a circular area of radius r, we shall have for the first quadrant,

$$P = \frac{1}{m^2 n^2} \int_0^r (m - x)\, dx \int_0^{\sqrt{r^2 - x^2}} (n - y)\, dy\,;$$

and for the whole circle,

$$P = \frac{4}{m^2 n^2} \int_0^r (m - x)\, dx \int_0^{\sqrt{r^2 - x^2}} (n - y)\, dy\,,$$

or

$$P = \frac{2r^2}{mn} \left[\frac{\pi}{2} - \frac{2r}{3m} - \frac{2r}{3n} + \frac{r^2}{4mn} \right]. \tag{5}$$

The probability that B will find the shoal depends upon the concurrence of two independent conditions whose separate probabilities are represented by equations (3) and (5), respectively, and is, therefore, equal to $P.\,p$, or

$$\frac{2r^2}{mn} \left\{ \frac{\pi}{2} - \frac{2r}{3m} - \frac{2r}{3n} + \frac{r^2}{4mn} \right\} \frac{(m - x)(n - y)}{m^2 n^2}\, \Delta x \Delta y\,.$$

Integrating the two expressions which make up equation (3) between the limits x and $x + \Delta x$ and y and $y + \Delta y$, respectively, the above expression becomes :

$$\frac{2r^2}{mn}\left\{\frac{\pi}{2} = \frac{2r}{3m} - \frac{2r}{3n} + \frac{r^2}{4mn}\right\} \frac{\Delta x}{m}\left[1 - \frac{2x + \Delta x}{2m}\right] \frac{\Delta y}{n}\left[1 - \frac{2y + \Delta y}{2n}\right],$$

which, for $r = 1$ mile, $x = 2$ miles, $y = 2$ miles, $m = 10$ miles, $n = 10$ miles, and Δx and Δy each equal to 1 mile, becomes $\frac{1}{6173}$. That is, under the conditions stated, B would stand one chance in 6173 of finding the shoal.

U. S. Hydrographic Office, *Nov.* 12, 1895.